Use integrating factor to solve:

a) $\dot{x} + kx = 1$

b) $\dot{x} + kx = e^{-5t}$ (for $k \neq 5$ and $k = 5$)

c) Use superposition to solve

$$\dot{x} + kx = 4 + 7e^{-5t}$$

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a) $\dot{x} + kx = 1 \quad - ①$

$$\frac{d}{dt}(xu) = \dot{x}u + x\dot{u}$$

$① \times u: \quad u\dot{x} + kux = u$

$$\Rightarrow u\dot{x} + \dot{u}x = u\dot{x} + kux$$

$$\dot{u}x = kux$$

$$\dot{u} = ku$$

$$\int \frac{1}{u}\frac{du}{dt}\,dt = \int k\,dt$$

$$\ln|u| = kt + c_1$$

$$u = \pm e^{kt} \cdot e^{c_1}$$

$$= Ce^{kt}$$

$\Rightarrow$ Let $u = e^{kt}$.

$$\frac{d}{dt}(ux) = u$$

$$\Rightarrow \int \frac{d}{dt}(ux)\,dt = \int u\,dt$$

$$\int \frac{d}{dt}(e^{kt}x)\,dt = \int e^{kt}\,dt$$

$$e^{kt}x = \frac{e^{kt}}{k} + C$$

$$x = \frac{1}{k} + \frac{C}{e^{kt}}$$

b) $\dot{x} + kx = e^{-5t}$

$u\dot{x} + kux = ue^{-5t}$

$u = Ce^{kt}$

$\Rightarrow$ Let $u = e^{kt}$.

$$\int \frac{d}{dt}(e^{kt}x) = \int e^{kt} \cdot e^{-5t}$$

$$\Rightarrow e^{kt}x = \frac{e^{(k-5)t}}{k-5} + C$$

$$\therefore x = \frac{e^{-5t}}{k-5} + \frac{C}{e^{kt}}$$

(when $k \neq 5$)

when $k = 5$,

$$\int \frac{d}{dt}(e^{5t} \cdot x) = \int e^{5t} \cdot e^{-5t}$$

$$e^{5t} \cdot x = t + C$$

$$x = \frac{t + C}{e^{5t}}$$

c) $\dot{x} + kx = 4 + 7e^{-5t}$

$\dot{x} + kx = 4$

$\Rightarrow \frac{d}{dt}(ux) = 4u, \quad u = e^{kt}$

$\Rightarrow ux = \frac{4e^{kt}}{k} + C$

$x = \frac{4}{k} + \frac{c}{e^{kt}}$

$\dot{x} + kx = 7e^{-5t}$

$\Rightarrow x = \frac{7e^{-5t}}{k-5} + \frac{c}{e^{kt}}$

$(k \neq 5)$

$x = \frac{7t + c}{e^{5t}}$

By superposition,

$\therefore x = \frac{4}{k} + \frac{2c}{e^{kt}} + \frac{7e^{-5t}}{k-5}$, when $k \neq 5$

$x = \frac{4}{5} + \frac{2c}{e^{5t}} + \frac{7t}{e^{5t}}$, when $k = 5$